Engaging Tasks + Productive Struggle = Math Success

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The Problems

The Band Concert Problem

The third-grade class is responsible for setting up the chairs for their spring band concert. In preparation, they need to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area. The class needs to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle. How many chairs does the school's engineer need to retrieve from the central storage area?

How might third grade students approach this problem?

Pay It Forward

In the movie "Pay it Forward", a student, Trevor, comes up with an idea that he thought could change the world. He decides to do a good deed for three people and then each of the three people would do a good deed for three more people and so on. He believed that before long there would be good things happening to billions of people. At stage 1 of the process, Trevor completes three good deeds. How does the number of good deeds grow from stage to stage? How many good deeds would be completed at stage 5? Describe a function that would model the Pay It Forward process at *any* stage.

How might students in an Algebra 1 or Integrated Mathematics 1 course approach this problem?

Exploring Representations for Multiplication
The Case of Mr. Harris and the Band Concert Task

Mr. Harris wanted his third-grade students to understand the structure of multiplication and decided to develop a task that would allow students to explore multiplication as equal groups through a familiar context—the upcoming spring band concert. He thought that the Band Concert Task (shown below) would prompt students to make or draw arrays and provide an opportunity to build conceptual understanding toward fluency in multiplying one-digit whole numbers by multiples of 10 using strategies based on place value and properties of operations—all key aspects of the standards for third grade students. He felt that the task aligned well with his math goals for the lesson and supported progress along math learning progressions, had multiple entry points, would provide opportunities for mathematical discourse, and it would challenge his students. As students worked on the task he would be looking for evidence that his students could identify the number of equal groups and the size of each group within visual or physical representations, such as collections or arrays, and connect these representations to multiplication equations.

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Mr. Harris began the lesson by asking students to consider how they might represent the problem. "Before you begin working on the task, think about a representation you might want to use and why, and then turn and share your ideas with a partner." The class held a short conversation sharing their suggestions, such as using cubes or drawing a picture. Then the students began working individually on the task.

As Mr. Harris made his way around the classroom, he noticed many students drawing pictures. Some students struggled to organize the information, particularly those who tried to represent each individual chair. He prompted these students to pause and review their work by asking, "So, tell me about your picture. How does it show the setup of the chairs for the band concert?" Other students used symbolic approaches, such as repeated addition or partial products, and a few students chose to use cubes or grid paper. He made note of the various approaches so he could decide which students he wanted to present their work, and in which order, later during the whole class discussion.

In planning for the lesson, Mr. Harris prepared key questions that he could use to press students to consider critical features of their representations related to the structure of multiplication. As the students worked, he often asked: "How does your drawing show the seven rows?" "How does your drawing show that there are 20 chairs in each row?" "Why are you adding all those twenties?" "How many twenties are you adding and why?"

He also noticed a few students changed representations as they worked. Dominic started to draw tally marks, but switched to using a table. When Mr. Harris asked her why, she explained she got tired of making all those marks. Similarly, Jamal started to build an array with cubes, but then switched to drawing an array. Their initial attempts were valuable, if not essential, in helping each of these students make sense of the situation.

Before holding a whole class discussion, Mr. Harris asked the students to find a classmate who had used a different representation and directed them to take turns explaining and comparing their work, as well as their solutions. He encouraged them to also consider how their representations were similar and different. For example, Jasmine who had drawn a diagram compared her work with Kenneth who had used equations (see reverse for copies of their work). Jasmine noted that they had gotten the same answer and Kenneth said they both had the number 20 written down seven times. Molly, in particular, was a student who benefited from this sharing process because she was able to acknowledge how confused she had gotten in drawing all those squares (see reverse side) and had lost track of her counting. Her partner helped her mark off the chairs in each row in groups of ten and recount them. The teacher repeated this process once more as students found another classmate and held another sharing and comparing session.

Written by DeAnn Huinker (University of Wisconsin-Milwaukee), drawing on experiences with teachers and students in the Milwaukee area. This case is intended to support the Guiding Principle for Teaching and Learning in *Principles to Actions: Ensuring Mathematical Success for All* (Reston, Va.: National Council of Teachers of Mathematics 2014).

During the whole class discussion, Mr. Harris asked the presenting students to explain what they had done and why and to answer questions posed by their peers. He asked Jasmine to present first since her diagram accurately modeled the situation and it would likely be accessible to all students. Kenneth went next as his approach was similar to Jasmine's but without the diagram. Both clearly showed the number 20 written seven times. Then Teresa presented. Her approach allowed the class to discuss how skip counting by twenties was related to the task and to multiplication, a connection not apparent for many students. Below is an excerpt from this discussion.

Mr. H: So, Teresa skip counted by twenties. How does this relate to the Band Concert situation?

Connor: She counted seven times like she wrote on her paper.

Mr. H: I'm not sure I understand, can someone add on to what Connor was saying?

Grace: Well each time she counted it was like adding 20 more chairs, just like what Kenneth did.

Mr. H: Do others agree with what Grace is saying? Can someone explain it in their own words?

Mason: Yeah, the numbers on top are like the 7 rows and the numbers on the bottom are the total number of chairs for that many rows.

Mr. H: This is interesting. So what does the number 100 mean under the 5?

Mason: It means that altogether five rows have 100 total chairs, since there are 20 chairs in each row.

Mr. H: Then what does the 140 mean?

Mason: It means that seven rows would have a total of 140 chairs.

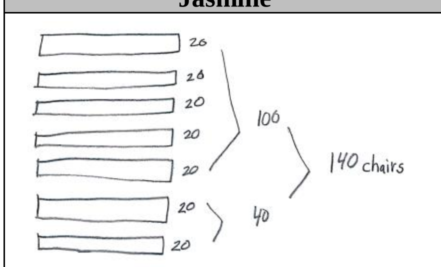
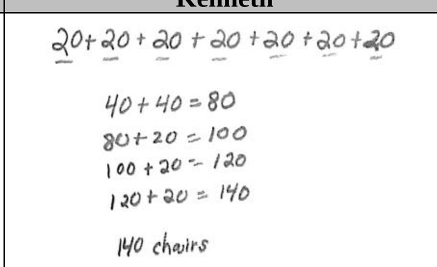
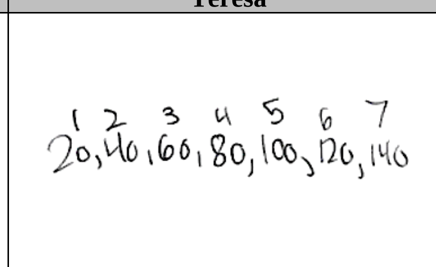
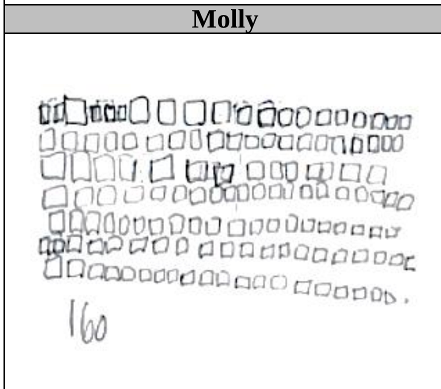
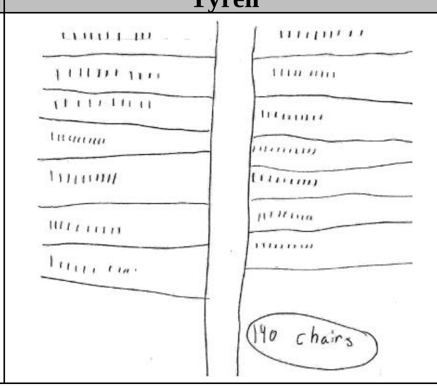
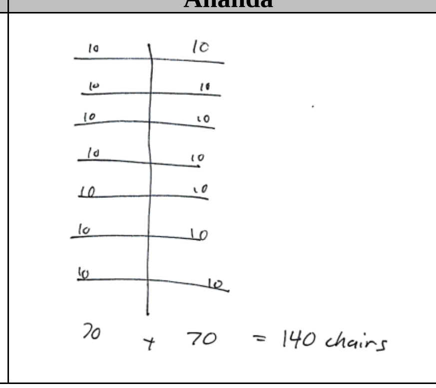
[Mr. Harris paused to write this equation on the board: 7 x 20 = 140.]

Mr. H: Some of you wrote this equation on your papers. How does this equation relate to each of the strategies that we have discussed so far? Turn and talk to a partner about this equation.

[After a few minutes, the whole class discussion continued and Grace shared what she talked about with her partner.]

Grace: Well, we talked about how the 7 means seven rows like Jasmine showed in her picture and how Teresa showed. And the 20 is the number of chairs that go in each row like Jasmine showed, and like how Kenneth wrote down. Teresa didn't write down all those twenties but we know she counted by twenty.

Toward the end of the lesson, Mr. Harris had Tyrell and Ananda present their representations because they considered the aisle and worked with tens rather than with twenties. After giving the students a chance to turn and talk with a partner, he asked them to respond in writing whether it was okay to represent and solve the task using either of these approaches and to justify their answers. He knew this informal experience with the distributive property would be important in subsequent lessons and the student writing would provide him with some insight into whether or not his students understood that quantities could be decomposed as a strategy in solving multiplication problems.

Jasmine	**Kenneth**	**Teresa**
		
Molly	**Tyrell**	**Ananda**
		

Exploring Exponential Relationships: *The Case of Ms. Culver*[1]

Ms. Culver wanted her students to understand that exponential functions grow by equal factors over equal intervals and that in the general equation $y = b^x$, the exponent (x) tells you how many times to use the base (b) as a factor. She also wanted students to see the different ways that the function could be represented and connected. She selected the Pay It Forward task because it provided a context that would help students in making sense of the situation, it could be model in several ways (i.e., diagram, table, graph, and equation), and it would challenge students to think and reason.

> In the movie "Pay It Forward", a student, Trevor, comes up with an idea that he thought could change the world. He decides to do a good deed for three people and then each of the three people would do a good deed for three more people and so on. He believed that before long there would be good things happening to billions of people. At stage 1 of the process, Trevor completes three good deeds. How does the number of good deeds grow from stage to stage? How many good deeds would be completed at stage 5? Describe a function that would model the Pay It Forward process at *any* stage.

Ms. Culver began the lesson by telling students to find a function that models the Pay It Forward process by any means necessary and that they could use any of the tools that were available in the classroom (e.g., graph paper, chart paper, colored pencils, markers, rulers, graphing calculators). As students began working in their groups, Ms. Culver walked around the room stopping at different groups to listen in on their conversations and to ask questions as needed (e.g., How did you get that? How do the number of good deeds increase at each stage? How do you know?). When students struggled to figure out what to do she encouraged them to try to visually represent what was happening at the first few stages and then to look for a pattern to see if there was a way to predict the way in which the number of deeds would increase in subsequent stages.

As she made her way around the room Ms. Culver also made note of the strategies students were using (see reverse side) so she could decide which groups she wanted to have present their work. She decided to have the strategies presented in the following sequence. Each presenting group would be expected to explain what they did and why and to answer questions posed by their peers. Group 4 would present their work first since their diagram accurately modeled the situation and would be accessible to all students. Group 3 would go next because their table summarized numerically what the diagram showed visually and made explicit the stage number, the number of deeds, and the fact that each stage involved multiplying by another 3. Groups 1 and 2 would then present their equations, one after the other. At this point Ms. Culver decided that she would give students 5 minutes to consider the two equations and decide which one they thought best modeled the situation and why.

Below is an excerpt from the discussion that took place after students in the class discussed the two equations that had been presented in their small groups.

Ms. C.: So who thinks that the equation $y = 3x$ best models the situation? Who thinks that the equation $y = 3^x$ best models the situation? *(Students raise their hands in response to each question.)*

Ms. C.: Can someone explain why $y = 3x$ is the best choice? Missy, can you explain how you were thinking about this?

Missy: Well Group 1 said that at every stage there are three times as many deeds as the one that came before it. That is what my group (4) found too when we drew the diagram. So the "3x" says that it is three times more."

Ms. C.: Does everyone agree with what Missy in saying? *(Lots of heads are shaking back and forth indicating disagreement.)* Darrell, why do you disagree with Missy?

Darrell: I agree that each stage has three times more good deeds that the previous stage, I just don't think that $y = 3x$ says that. If x is the stage number like we said then the equation says that the number of deeds is three times the stage number – not three times the

[1] This case, written by Margaret Smith (University of Pittsburgh), is based on a lesson planned and taught by Michael Betler, a student completing his secondary mathematics certification and MAT degree at the University of Pittsburgh during the 2013-2014 school year.

number of deeds in the previous stage. So the number of deeds is only 3 more not 3 times more.

Ms. C.: Other comments?

Kara: I agree with Darrell. y = 3x works for stage 1 but it doesn't work for the other stages. If we look at the diagram it shows that stage 2 has 9 good deeds. But, If you use the equation you get 6 not 9. So it can't be right.

Chris: y = 3x is linear. If this function were linear then the first stage would be three, the next stage would be 6, then the next stage would be 9. This function can't be linear – it gets really big fast. There isn't a constant rate of change.

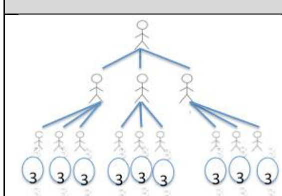
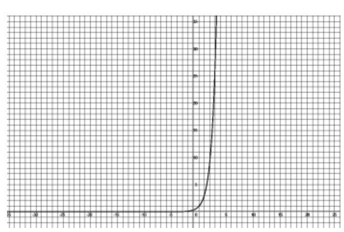
Ms. C.: So let's take another look at Group 3's poster. Does the middle column help explain what is going on? Devon?

Devon: Yeah. They show that each stage has 3 times more deeds that the previous one. For each stage there is one more three that gets multiplied. That makes the new one three times more than the previous one.

Angela: So that is why I think $y = 3^x$ best models the situation. Stage 1 had 3 good deeds, stage 2 people had three each doing three deeds so that is 3^2, stage 3 had 9 people (3^2) each doing 3 good deeds, so that is 3^3. The x tells how many 3's are being multiplied. So as the stage number increases by 1 the number of deeds gets three times larger.

Ms. C.: If we keep multiplying by another three like Angela described, it is going to get big really fast like Chris said. Chris also said it couldn't be linear, so take a minute and think about what the graph would look like.

At this point Ms. Culver asked Group 5 to share their graph and proceeded to engage the class in a discussion of what the domain of the function should be given the context of the problem. The lesson concluded with Ms. Culver telling the students that the function they had created was called exponential and explaining that exponential functions are written in the form of $y = b^x$. She told students that in the five minutes that remained in class they needed to individually explain in writing how the equation related to the diagram, the table, the graph and the problem context. She thought that this would give her some insight regarding what students understood about exponential functions and the relationship between the different ways the function could be represented.

<table>
<tr><th>Group 1
(equation - incorrect)</th><th>Group 2
(table like Group 6 & 7 and equation)</th><th>Group 3
(diagram like Group 4 and table)</th></tr>
<tr><td>y = 3x
At every stage there are three times as many good deeds as there were in the previous stage.</td><td>$y = 3^x$</td><td><table>
<tr><th>x (stages)</th><th></th><th>y (deeds)</th></tr>
<tr><td>1</td><td>3</td><td>3</td></tr>
<tr><td>2</td><td>3 x 3</td><td>9</td></tr>
<tr><td>3</td><td>3 x 3 x 3</td><td>27</td></tr>
<tr><td>4</td><td>3 x 3 x 3 x 3</td><td>81</td></tr>
<tr><td>5</td><td>3 x 3 x 3 x 3 x 3</td><td>243</td></tr>
</table></td></tr>
<tr><th>Group 4
(diagram)</th><th>Group 5
(table like Group 6 & 7 and graph)</th><th>Groups 6 and 7
(table)</th></tr>
<tr><td>
So the next stage will be 3 times the number there in the current stage so 27 x 3. It is too many to draw. You keep multiplying by 3.</td><td></td><td><table>
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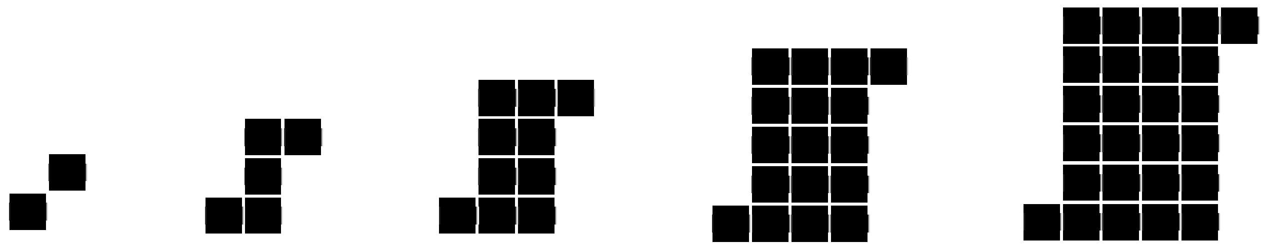
The S Pattern Task[1]



1 2 3 4 5

1. What patterns do you notice in the set of figures?

2. How many square tiles are in figure 7? Write a description that could be used to determine the shape of and total number of square tiles in figure 7. Your description should be clear enough so that another person could read it and use it to think about another figure.

3. Determine an equation for the total number of squares in any figure. Explain your rule and show how it relates to the visual diagram of the figures.

4. Find a second way to describe the pattern and write the equation that matches the description. Compare the two equations and show in the visual representation how one equation is equivalent to the other.

5. If you knew that a figure had 9802 squares tiles in it, how could you determine the figure number? Explain.

6. Does the pattern describe a linear relationship between the figure number and the total number of squares? Why or why not?

Two Teachers' Responses to Students' Struggles to Solve a Multi-step Word Problem Involving Fractions

Ms. Flahive and Ms. Ramirez teach fifth grade and plan their lessons collaboratively. Their current instructional unit focuses on fractions. They have selected the Shopping Trip task shown below because they think it will be accessible to their students, yet provoke some struggle and challenge, since a solution pathway is not straightforward. The mathematics goal for students is to draw on and apply their understanding of how to build non-unit fractions from unit fractions and to use visual representations to solve a multi-step word problem:

> **Shopping Trip Task**
> Joseph went to the mall with his friends to spend the money that he had received for his birthday. When he got home he still had $24 remaining. He had spent $^{3}/_{5}$ of his birthday money at the mall on video games and food. How much money did he spend?
> How much money had he received for his birthday?

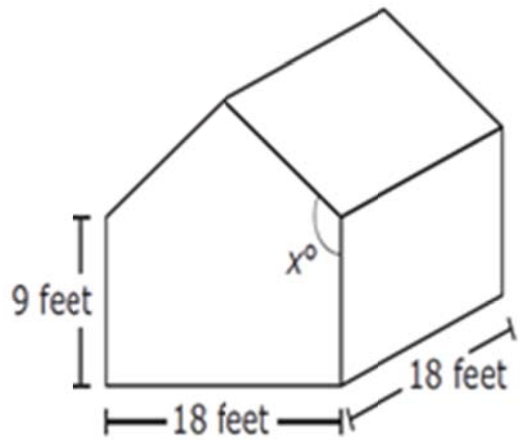
When Ms. Flahive and Ms. Ramirez present the problem in their classrooms, both teachers see students struggling to get started. Some students in both classrooms immediately raise their hands, saying, "I don't get it," or "I don't know what to do."

Ms. Flahive is very directive in her response to her students. She tells them to draw a rectangle and shows them how to divide it into fifths to represent what Joseph had spent and what he had left. She then guides her students step by step until they have labeled each one-fifth of the rectangle as worth $12, as shown below. Finally, she tells the students to use the information in the diagram to figure out the answers to the questions.

Ms. Ramirez approaches her students' struggles very differently. After she sees them struggling, she has them stop working on the problem and asks all the students to write down two things that they know about the problem and one thing that they wish they knew because it would help them make progress in solving the problem. Then Ms. Ramirez initiates a short class discussion in which several ideas are offered for what to do next. Suggestions include drawing a tape diagram or number line showing fifths, or just picking a number, such as $50 and acting it out through trial and error. Ms. Ramirez encourages the students to consider the various ideas that have been shared as they continue working on the task.

Principles to Actions, p. 51

Designing a Shed



The base of the shed will be a square measuring 18 feet by 18 feet. The height of the rectangular sides will be 9 feet. The measure of the angle made by the roof with the side of the shed can vary and is labeled as X°. Different roof angles create different surface areas of the roof. The surface area of the roof will determine the number of roofing shingles needed in constructing the shed. To meet drainage requirements, the roof angle must be at least 117°.